ON WIGNER DECOMPOSITION OF LORENTZ MATRICES

Roberto Vergara Caffarelli

Dipartimento di Fisica dell'Università di Pisa
and
Istituto Nazionale di Fisica Nucleare

Pisa - Italia

ABSTRACT

We derive a simple formula for the Wigner decomposition of any matrix of the vector representation of the orthochronous proper Lorentz group into a product of a rotation and a boost. When this formula is applied to the product of two boosts, in the case of non-parallel velocities, the matrix of the associate rotation follows in a very simple way. The relativistic composition of velocities is also easily found.

1. INTRODUCTION

In this paper we derive a simple formula for the "Wigner decomposition" of any matrix Λ of the vector representation of the orthochronous proper Lorentz group into a product of a rotation $\Omega(\Lambda)$, which we name here the *associate spatial rotation* , and a boost $\Pi(\Lambda)$, named the *associate pure Lorentz transformation* :

$$\Lambda = \Omega\,\Pi \tag{1}$$

The uniqueness of the decomposition is true also in the reverse order [1]. When this formula is applied to the product of two boosts, in the case of non-parallel velocities, the matrix of the associate rotation follows in a very simple way. The relativistic composition of velocities is also easily found. The paper is self-contained and suggests an approach to studying Lorentz matrices. This pedagogical purpose leads us to derive in detail the formulae. It is unnecessary any representation space for the Lorentz group other than space-time: hence, all results will be expressed directly as funtions of the matrix elements $\Lambda_{\alpha\beta}$. It is surprising that more sophisticated methods are usually involved and explicit formulae are missing in current literature.

Let us introduce the orthochronous proper Lorentz group as the group of 4x4 matrices characterized by the following requirements [2]:

$$\Lambda^T \eta\, \Lambda = \eta \tag{2}$$

$$\Lambda_{oo} \geq 1 \tag{3}$$

$$\det \Lambda = 1 \tag{4}$$

η is the Minkowski metric in an inertial system of cartesian coordinates, with matrix elements

$$\eta_{\alpha\beta} = \delta_{\alpha\beta} - 2\,\delta_{\alpha o}\,\delta_{\beta o} \tag{5}$$

In our notation Greek indices will always run over the four space-time values 0, 1, 2, 3 ; Latin indices only run over the spatial coordinate labels 1, 2, 3; repeated indices are summed over their respective ranges unless otherwise indicated. We use units in which the speed of light is unity.

Condition (2) is a consequence of the invariance of $ds^2 = \eta_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$ and it is fully equivalent to the following "orthogonality conditions" in Minkowski space-time

$$\Lambda_{i\alpha}\Lambda_{i\beta} - \Lambda_{o\alpha}\Lambda_{o\beta} = \delta_{\alpha\beta} - 2\,\delta_{o\alpha}\delta_{o\beta} \tag{6}$$

Using (2), the inverse matrix Λ^{-1} can be easily espressed in terms of the matrix elements of Λ :

$$\Lambda^{-1} = \eta\ \Lambda^{T}\,\eta. \tag{7}$$

The inverse matrix Λ^{-1} verify (2)

$$(\Lambda^{-1})^{T}\,\eta\ \Lambda^{-1} = \eta \tag{7'}$$

By eqs. (7) and (7') we see that also Λ^{T} is a Lorentz transformation

$$\Lambda\,\eta\ \Lambda^{T} = \eta$$

or, explicitly:

$$\Lambda_{\alpha i}\Lambda_{\beta i} - \Lambda_{\alpha o}\Lambda_{\beta o} = \delta_{\alpha\beta} - 2\,\delta_{o\alpha}\delta_{o\beta} \tag{8}$$

2. THE DECOMPOSITION

If $\Lambda_{oo} = 1$, the immediate consequence of (6) and (8) is $\Lambda_{io} = 0$ and $\Lambda_{oi} = 0$ for i = 1, 2, 3. The matrix Λ in this case describes a *spatial rotation*, which verifies the relation

$$\Lambda^{T}\,\Lambda = 1 \tag{9}$$

It is appropriate at this point to prove the following theorem:
Theorem 1.
The matrix Ω is a spatial rotation if the matrix elements $\Omega_{\alpha\beta}$ are obtained from the Lorentz transformation Λ in the following manner:

$$\Omega_{mn} = \Lambda_{mn} - \frac{\Lambda_{mo}\ \Lambda_{on}}{\Lambda_{oo} + 1} \tag{10}$$

$$\Omega_{oo} = 1 \tag{11}$$

$$\Omega_{om} = \Omega_{mo} = 0 \tag{12}$$

Ω will be called *the associate rotation* of Λ.

The proof is given in appendix A.

We introduce now the matrix Π which elements $\Pi_{\alpha\beta}$ are obtained from Λ in the following manner

$$\Pi_{oo} = \Lambda_{oo} \tag{13}$$

$$\Pi_{on} = \Pi_{no} = \Lambda_{on} \tag{14}$$

$$\Pi_{mn} = \delta_{mn} + \Lambda_{om}\,\Lambda_{on}\,(\Lambda_{oo} + 1)^{-1} \tag{15}$$

One easily shows the following

Theorem 2.

Any matrix Λ of the orthochronous proper Lorentz group is the product of the associated rotation Ω and the matrix Π.

$$\Omega\Pi = \Lambda \tag{16}$$

The proof is given in appendix B

Because of the group properties, Π is a Lorentz matrix as Λ and Ω.
We call Π *the pure Lorentz transformation associated to* Λ.
The rotation angle θ can be obtained from

$$\mathrm{tr}\,\Omega = 2(1 + \cos\theta) = 1 + \Lambda_{ss} - \Lambda_{so}\,\Lambda_{os}\,(\Lambda_{oo} + 1)^{-1}$$

Thus θ verifies the simple relation

$$2\cos\theta = \Lambda_{ss} - 1 - \Lambda_{so}\,\Lambda_{os}\,(\Lambda_{oo} + 1)^{-1} \tag{17}$$

3. THE DEPENDENCE OF $\Lambda_{\mu\nu}$ FROM VELOCITY AND ROTATION

If a point moves with velocity v_i in the system S, then it will be at rest in the system S' obtained from S with the boost $\Pi(v_i)$.
From

$$\delta^{\mu o} = dx'^{\mu}/d\tau = \Lambda_{\mu\nu}\, dx^{\nu}/d\tau$$

we find that $0 = \Lambda_{i\nu}\, dx^{\nu}$, or

$$v_i = dx^i/dx^o = -\Lambda_{oi} / \Lambda_{oo} \tag{18}$$

Let u_i be [3]

$$u_i = \Omega_{im}\, v_m = -\Lambda_{io}/\Lambda_{oo} \tag{19}$$

We use in the following the natural notation

$$v^2 = v_1^2 + v_2^2 + v_3^2$$

$$u^2 = u_1^2 + u_2^2 + u_3^2$$

From eqs. (A7), (B7) , or directly from eq. (19), we have

$$v^2 = u^2 \tag{20}$$

Using eqs. (18) we obtain the usual explicit representation of matrix Π in terms of velocity

$$\Pi(v_i) = \begin{vmatrix} \gamma & -\gamma v_1 & -\gamma v_2 & -\gamma v_3 \\ -\gamma v_1 & & & \\ -\gamma v_2 & & \delta_{mn} + v_m v_n(\gamma - 1)\, v^{-2} & \\ -\gamma v_3 & & & \end{vmatrix} \tag{21}$$

where

$$\gamma = (1 - v^2)^{-1/2} \tag{22}$$

The matrix elements of Ω may be conveniently expressed as

$$\Omega_{mn} = \Lambda_{mn} - (\gamma - 1)\, u_m v_n v^{-2} \tag{23}$$

Conversely, any matrix of the orthochronous proper Lorentz group can be expressed as a function of the v_i and u_i components of velocity and of matrix elements of the rotation Ω (as determined in the original frame) in the following way

$$\Lambda(v_i, \Omega) = \begin{vmatrix} \gamma & -\gamma v_1 \quad -\gamma v_2 \quad -\gamma v_3 \\ -\gamma u_1 & \\ -\gamma u_2 & \Omega_{mn} + u_m v_n (\gamma - 1)\, v^{-2} \\ -\gamma u_3 & \end{vmatrix} \tag{24}$$

One may ask why $\Pi(v_i)$ is a pure Lorentz trasformation, i.e. if we have ruled out from Π any additional rotation. To see that, it is helpful to think of rotations as motions occurring in space-like two-planes, with only one fixed point.

We exclude rotations showing that any point of a space-like plane a**A** + b**B** is a fixed point under Π.

Let **A** = $(0, a_1, a_2, a_3)$ and **B** = $(0, b_1, b_2, b_3)$ be two orthogonal space-like four vectors

$$\eta_{\mu\nu} A^\mu B^\nu = 0 \tag{25}$$

Since $-\Lambda_{o\mu} = -\Pi_{o\mu}$ are the *covariant* components of the four-velocity $v_\mu = \eta_{\mu\nu} dx^\nu/d\tau$, which is a time-like vector, if $v_i a_i = v_i b_i = 0$, then **A** and **B** can be choosen orthogonal to $dx^\nu/d\tau$, and one gets

$$\Pi_{o\mu} A^\mu = \Pi_{o\mu} B^\mu = 0 \tag{26}$$

$$\Pi\, \mathbf{A} = \mathbf{A} \tag{27}$$

$$\Pi\, \mathbf{B} = \mathbf{B} \tag{28}$$

From eq. (24) one can easily see that any symmetric Lorentz matrix $\Lambda(v_i, \Omega)$ is a boost or a boost followed by a π-rotation about **v**, since $\Omega = \Omega^{-1}$.

4. PRODUCT OF TWO PURE LORENTZ TRANSFORMATIONS

It is easy to show that the product of two boosts

$$\Pi(u_i)\,\Pi(k_i) = \Lambda(v_i, \Omega)$$

is a boost only if the velocities $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{k} = (k_1, k_2, k_3)$ are parallel. The relevant matrix elements of this product are

$$\Lambda_{oo} = \gamma_u \gamma_k \left(1 + \mathbf{u}\cdot\mathbf{k}\right) \tag{29}$$

$$\Lambda_{om} = -\gamma_u \left[\gamma_k k_m + u_m + \mathbf{u}\cdot\mathbf{k}\left(\gamma_k - 1\right)k_m k^{-2}\right] \tag{30}$$

$$\Lambda_{mo} = -\gamma_k \left[\gamma_u u_m + k_m + \mathbf{u}\cdot\mathbf{k}\left(\gamma_u - 1\right)u_m u^{-2}\right] \tag{31}$$

$$\Lambda_{mn} = \gamma_u \gamma_k u_m k_n + \delta_{mn} + k_m k_n \left(\gamma_k - 1\right) k^{-2} + u_m u_n \left(\gamma_u - 1\right)u^{-2} + u_m k_n \,\mathbf{u}\cdot\mathbf{k}\; u^{-2} k^{-2} \left(\gamma_u - 1\right)\left(\gamma_k - 1\right) \tag{32}$$

By using the identities

$$\left(\gamma_u - 1\right) u^{-2} = \gamma_u^2 \left(\gamma_u + 1\right)^{-1} \tag{33}$$

$$\left(\gamma_k - 1\right) k^{-2} = \gamma_k^2 \left(\gamma_k + 1\right)^{-1} \tag{34}$$

and one can write the above expressions (30) and (31) as the components of a vector combination:

$$\Lambda_{om} = -\gamma_u \gamma_k \left[\, \mathbf{k} + \mathbf{u} + \gamma_k \left(\gamma_k + 1\right)^{-1} \mathbf{k} \times (\mathbf{k} \times \mathbf{u}) \,\right]_m \tag{30'}$$

$$\Lambda_{mo} = -\gamma_k \gamma_u \left[\, \mathbf{u} + \mathbf{k} + \gamma_u \left(\gamma_u + 1\right)^{-1} \mathbf{u} \times (\mathbf{u} \times \mathbf{k}) \,\right]_m \tag{31'}$$

From eq. (29) we obtain

$$\gamma_v = \gamma_h \gamma_k \left(1 + \mathbf{u}\cdot\mathbf{k}\right) \tag{35}$$

and [4]

$$v^2 = 1 - \left(1 - u^2\right)\left(1 - k^2\right)\left(1 + \mathbf{u}\cdot\mathbf{k}\right)^{-2} \tag{36}$$

If $\Lambda(v_i, \Omega)$ is a boost, from the property $\Lambda_{oi} = \Lambda_{io}$ one must equate (30') and (31')

$$\gamma_k \left(\gamma_k + 1\right)^{-1} \mathbf{k} \times (\mathbf{k} \times \mathbf{u}) = \gamma_u \left(\gamma_u + 1\right)^{-1} \mathbf{u} \times (\mathbf{u} \times \mathbf{k}) \tag{37}$$

It is obvious from eq. (37) that **k** and **u** must be parallel. Note that the product is commutative for collinear boosts.
In any case from eq. (18) we obtain the *composition law of the velocities*

$$\mathbf{v} = \frac{\mathbf{k} + \mathbf{u} + \gamma_k \left(\gamma_k + 1\right)^{-1} \mathbf{k} \times (\mathbf{k} \times \mathbf{u})}{\left(1 + \mathbf{u}\cdot\mathbf{k}\right)} \tag{38}$$

v is the velocity of a point in the laboratory and **u** is the velocity of the same point in a frame that moves with velocity **k** relative to the laboratory.
If **k** and **u** are not parallel, the matrix elements Ω_{mn} of the associate rotation (10) are given by

$$\begin{aligned}
\Omega_{mn} = \; & \delta_{mn} + \gamma_u \gamma_k u_m k_n + k_m k_n \left(\gamma_k - 1\right) k^{-2} \\
& + u_m u_n \left(\gamma_u - 1\right) u^{-2} + u_m k_n \, \mathbf{u}\cdot\mathbf{k} \left(\gamma_u - 1\right)\left(\gamma_k - 1\right) u^{-2} k^{-2} \\
& - \gamma_u \gamma_k \left[\gamma_u \gamma_k \left(1 + \mathbf{u}\cdot\mathbf{k}\right) + 1\right]^{-1} \left\{\left[\gamma_u u_m + k_m + \mathbf{u}\cdot\mathbf{k}\left(\gamma_u - 1\right) u_m u^{-2}\right]\right. \\
& \left.\left[\gamma_k k_n + u_n + \mathbf{u}\cdot\mathbf{k}\left(\gamma_k - 1\right) k_n k^{-2}\right]\right\}
\end{aligned} \tag{39}$$

We may obtain a more compact form of (39) by straightforward computation, (carried forward in appendix C); the matrix elements of the associate rotation are given more simply by

$$\Omega_{mn} = \delta_{mn} - \frac{(\gamma_u - 1)(\gamma_k - 1)}{\gamma_u \gamma_k (1 + u\cdot k) + 1} \left[\frac{k_m k_n}{k^2} + \frac{u_m u_n}{u^2} \right.$$

$$\left. - 2 \frac{u\cdot k}{u^2 k^2} u_m k_n \right] + \frac{\gamma_u \gamma_k}{\gamma_u \gamma_k (1 + u\cdot k) + 1} (u_m k_n - k_m u_n) \quad (40)$$

It is obvious from eq. (40) that the vector (0, **u x k**) is a fixed vector under the rotation Ω, which can also be written as

$$\Omega_{mn} = \delta_{mn} - \frac{(\gamma_u - 1)(\gamma_k - 1)\, u^{-2} k^{-2}}{\gamma_u \gamma_k (1 + u\cdot k) + 1} \left\{ [u \times (k \times u)]_m k_n \right.$$

$$\left. + u_m [k \times (u \times k)]_n \right\} + \frac{\gamma_u \gamma_k}{\gamma_u \gamma_k (1 + u\cdot k) + 1} (u_m k_n - k_m u_n) \quad (41)$$

The rotation angle is given by

$$\mathrm{tr}\,\Omega = 2(1 + \cos\theta) = 4 - 2 \frac{(\gamma_u - 1)(\gamma_k - 1)}{\gamma_u \gamma_k (1 + u\cdot k) + 1} \left[1 - \frac{(u\cdot k)^2}{u^2 k^2} \right] \quad (42)$$

or

$$\cos\theta = 1 - \frac{(\gamma_u - 1)(\gamma_k - 1)}{\gamma_u \gamma_k (1 + u\cdot k) + 1} \left[1 - \frac{(u\cdot k)^2}{u^2 k^2} \right] \quad (43)$$

The angle θ is independent of the order of the product $\Pi(u_i)\Pi(k_j)$ or $\Pi(k_j)\Pi(u_i)$.

Let introduce the angle ϕ between **u** and **k** as **u·k** = uk cos ϕ, then

$$\cos\theta = 1 - \frac{(\gamma_u - 1)\,(\gamma_k - 1)}{\gamma_u\,\gamma_k\,(1 + uk\cos\Phi) + 1}\ \mathrm{sen}^2\,\phi \tag{44}$$

In case of collinear velocities, i.e. when $u = \lambda\, k$,

$$\Omega^m{}_n = \delta_{mn}$$

hence, we see directly that the product of two collinear transformations does not entail any rotation.
The product of Lorentz matrices without any restriction is discussed in appendix D.
The discussion, within this approach, of the Thomas precession will be included in a paper in preparation.

APPENDIX A

Theorem 1
If we construct from the matrix elements of a Lorentz matrix Λ the following 4x4 matrix Ω :

$$\Omega_{mn} = \Lambda_{mn} - \frac{\Lambda_{mo}\ \Lambda_{on}}{\Lambda_{oo} + 1} \tag{A1}$$

$$\Omega_{oo} = 1 \tag{A2}$$

$$\Omega_{on} = \Omega_{mo} = 0 \tag{A3}$$

Then Ω is a spatial rotation, i.e.

$$\Omega\,\Omega^T = \Omega^T\,\Omega = 1 \tag{A4}$$

Proof of theorem 1

It suffices to calculate $\Omega_{sm}\ \Omega_{sn}$.
We list the relations (6) used for computing the above matrix moltiplication:

$$\Lambda_{sm}\Lambda_{sn} = \delta_{mn} + \Lambda_{om}\Lambda_{on} \tag{A5}$$

$$\Lambda_{sm}\Lambda_{so} = \Lambda_{om}\Lambda_{oo} \tag{A6}$$

$$\Lambda_{so}\Lambda_{so} = \Lambda_{oo}\Lambda_{oo} - 1 \tag{A7}$$

thus, using (A5-A7) we have

$$\Omega_{sm}\Omega_{sn} = \left(\Lambda_{sm} - \frac{\Lambda_{so}\Lambda_{om}}{\Lambda_{oo}+1}\right)\left(\Lambda_{sn} - \frac{\Lambda_{so}\Lambda_{on}}{\Lambda_{oo}+1}\right)$$

$$= \Lambda_{sm}\Lambda_{sn} - (\Lambda_{sm}\Lambda_{so}\Lambda_{on} + \Lambda_{sn}\Lambda_{so}\Lambda_{om})(\Lambda_{oo}+1)^{-1}$$

$$+ \Lambda_{so}\Lambda_{om}\Lambda_{so}\Lambda_{on}(\Lambda_{oo}+1)^{-2}$$

$$= \delta_{mn} + \Lambda_{om}\Lambda_{on} - 2\Lambda_{om}\Lambda_{on}\Lambda_{oo}(\Lambda_{oo}+1)^{-1}$$

$$+ \Lambda_{om}\Lambda_{on}(\Lambda_{oo}\Lambda_{oo} - 1)(\Lambda_{oo}+1)^{-2} = \delta_{mn}$$

APPENDIX B

Theorem 2.

Any matrix Λ of the proper orthochronous Lorentz group is the product of the associated rotation Ω and the matrix Π defined in the following way:

$$\Pi_{oo} = \Lambda_{oo} \tag{B1}$$

$$\Pi_{on} = \Pi_{no} = \Lambda_{on} \tag{B2}$$

$$\Pi_{mn} = \delta_{mn} + \Lambda_{om}\Lambda_{on}(\Lambda_{oo}+1)^{-1} \tag{B3}$$

It suffices to prove the following two relations

$$\Omega_{ms} \Pi_{so} = \Lambda_{mo} \tag{B4}$$

$$\Omega_{ms} \Pi_{sn} = \Lambda_{mn} \tag{B5}$$

We can check (B4) by using eq. (8), or more explicitly

$$\Lambda_{ms} \Lambda_{os} = \Lambda_{mo} \Lambda_{oo} \tag{B6}$$

$$\Lambda_{os} \Lambda_{os} = \Lambda_{oo} \Lambda_{oo} - 1 \tag{B7}$$

as follows

$$\begin{aligned} \Omega_{ms} \Pi_{so} &= \Lambda_{ms} \Lambda_{os} - \Lambda_{mo} \Lambda_{os} \Lambda_{os} (\Lambda_{oo} + 1)^{-1} \\ &= \Lambda_{mo} \Lambda_{oo} - (\Lambda_{oo} \Lambda_{oo} - 1) \Lambda_{mo} (\Lambda_{oo} + 1)^{-1} \\ &= \Lambda_{mo} \Lambda_{oo} - (\Lambda_{oo} - 1) \Lambda_{mo} = \Lambda_{mo} \end{aligned}$$

Expression (B5) is checked by using eqs. (10), (15), (B6) and (B7)

$$\begin{aligned} \Omega_{ms}\Pi_{sn} = {} & (\Lambda_{ms} - \frac{\Lambda_{mo} \Lambda_{os}}{\Lambda_{oo} + 1})[\delta_{sn} + \Lambda_{os} \Lambda_{on} (\Lambda_{oo} + 1)^{-1}] \\ = {} & \Lambda_{mn} + (\Lambda_{ms} \Lambda_{os} \Lambda_{on} - \Lambda_{mo} \Lambda_{on})(\Lambda_{oo} + 1)^{-1} \\ & - \Lambda_{mo} \Lambda_{os} \Lambda_{os} \Lambda_{on} (\Lambda_{oo} + 1)^{-2} \\ = {} & \Lambda_{mn} + (\Lambda_{mo} \Lambda_{oo} \Lambda_{on} - \Lambda_{mo} \Lambda_{on})(\Lambda_{oo} + 1)^{-1} \\ & - \Lambda_{mo} \Lambda_{on} (\Lambda_{oo} \Lambda_{oo} - 1)(\Lambda_{oo} + 1)^{-2} \\ = {} & \Lambda_{mn} + (\Lambda_{mo} \Lambda_{oo} \Lambda_{on} - \Lambda_{mo} \Lambda_{on})(\Lambda_{oo} + 1)^{-1} \\ & - \Lambda_{mo} \Lambda_{on} (\Lambda_{oo} - 1)(\Lambda_{oo} + 1)^{-1} = \Lambda_{mn} \end{aligned}$$

APPENDIX C

We can write eq. (39) as follows

$$\Omega_{mn} = \delta_{mn} - \gamma_u \gamma_k [\gamma_u \gamma_k (1 + u \cdot k) + 1]^{-1} k_m u_n$$

$$+ A \left(k_m k_n k^{-2} + u_m u_n u^{-2} \right) + B u_m k_n \qquad \text{(C1)}$$

The coefficient A is easily evaluated (the asymmetry between u and k will disappear at the end)

$$A = \gamma_k - 1 - [\gamma_u \gamma_k (1 + u \cdot k) + 1]^{-1} [k^2 \gamma_u \gamma_k^2 + \gamma_u \gamma_k u \cdot k (\gamma_k - 1)]$$

$$= \gamma_k - 1 - [\gamma_u \gamma_k (1 + u \cdot k) + 1]^{-1} \{ [k^2 \gamma_u \gamma_k^2 - (\gamma_k - 1)(\gamma_u \gamma_k + 1)$$

$$+ (\gamma_k - 1) [\gamma_u \gamma_k (1 + u \cdot k) + 1] \}$$

$$= - [\gamma_u \gamma_k (1 + u \cdot k) + 1]^{-1} [(k^2 - 1) \gamma_u \gamma_k^2 - \gamma_k + 1 + \gamma_u \gamma_k]$$

$$= - (\gamma_u - 1)(\gamma_k - 1) [\gamma_u \gamma_k (1 + u \cdot k) + 1]^{-1} \qquad \text{(C2)}$$

The coefficient B is given by

$$B = \gamma_u \gamma_k + u \cdot k (\gamma_u - 1)(\gamma_k - 1) u^{-2} k^{-2}$$

$$- \frac{\gamma_u \gamma_k [\gamma_u + u \cdot k (\gamma_u - 1) u^{-2}] [\gamma_k + u \cdot k (\gamma_k - 1) k^{-2}]}{[\gamma_u \gamma_k (1 + u \cdot k) + 1]}$$

$$= \gamma_u \gamma_k + u \cdot k \gamma_u^2 \gamma_k^2 (\gamma_u + 1)^{-1} (\gamma_k + 1)^{-1}$$

$$- \frac{(\gamma_u \gamma_k)^2 [\gamma_u (1 + u \cdot k) + 1] [\gamma_k (1 + u \cdot k) + 1]}{(\gamma_u + 1)(\gamma_k + 1) [\gamma_u \gamma_k (1 + u \cdot k) + 1]} \qquad \text{(C3)}$$

As an intermediate step, it is useful to multiply the first and second term in eq. (C3) by the denominator of the last term.

$$(\gamma_u + 1)(\gamma_k + 1) [\gamma_u \gamma_k (1 + u \cdot k) + 1] \{ \gamma_u \gamma_k$$

$$+ u \cdot k \gamma_u^2 \gamma_k^2 (\gamma_u + 1)^{-1} (\gamma_k + 1)^{-1} \}$$

$$= \gamma_u \gamma_k (\gamma_u + 1)(\gamma_k + 1) + \gamma_u^2 \gamma_k^2 (\gamma_u + 1)(\gamma_k + 1)(1 + u\cdot k)$$

$$+ [\gamma_u \gamma_k (1 + u\cdot k) + 1]\, u\cdot k\, \gamma_u^2 \gamma_k^2$$

$$= \gamma_u \gamma_k (\gamma_u + 1)(\gamma_k + 1) + \gamma_u^2 \gamma_k^2 \{ \gamma_u \gamma_k (1 + u\cdot k) + \gamma_u (1 + u\cdot k)$$

$$+ \gamma_k (1 + u\cdot k) + (1 + u\cdot k) + [\gamma_u \gamma_k (1 + u\cdot k) + 1](1 + u\cdot k)$$

$$- \gamma_u \gamma_k (1 + u\cdot k) - 1 \}$$

$$= \gamma_u \gamma_k (\gamma_u + 1)(\gamma_k + 1)$$

$$+ \gamma_u^2 \gamma_k^2 \{ [\gamma_u (1 + u\cdot k) + 1][\gamma_k (1 + u\cdot k) + 1] + 2\, u\cdot k \} \qquad \text{(C4)}$$

By using eq. (C4) one has

$$B = \frac{[\gamma_u \gamma_k (\gamma_u + 1)(\gamma_k + 1) + 2\, u\cdot k\, \gamma_u^2 \gamma_k^2]}{(\gamma_u + 1)(\gamma_k + 1)[\gamma_u \gamma_k (1 + u\cdot k) + 1]} \qquad \text{(C5)}$$

and the matrix elements of the associate rotation are

$$\Omega_{mn} = \delta_{mn} - \frac{(\gamma_u - 1)(\gamma_k - 1)}{\gamma_u \gamma_k (1 + u\cdot k) + 1} \Big[\frac{k_m k_n}{k^2} + \frac{u_m u_n}{u^2}$$

$$- 2 \frac{u\cdot k}{u^2 k^2} u_m k_n \Big] + \frac{\gamma_u \gamma_k}{\gamma_u \gamma_k (1 + u\cdot k) + 1} (u_m k_n - k_m u_n) \qquad \text{(C6)}$$

APPENDIX D

It is straightforward to apply the decomposition (16) to the product of two Lorentz transformations $\Omega \Pi(u) = \Lambda(u, \Omega)$ and $\Phi \Pi(k) = \Lambda(k, \Phi)$, but it is instructive to give also a direct derivation. Inspection shows that in the case of the product of a pure Lorentz transformation with a rotation we have

$$\Pi(u)\, \Phi = \Phi\, \Pi(h) \qquad \text{(D1)}$$

where

$$\mathbf{h} = \Phi^{-1}\,\mathbf{u}. \tag{D2}$$

Consequently, we have

$$\Lambda(\mathbf{u},\Omega)\,\Lambda(\mathbf{k},\Phi) = \Omega\,\Pi(\mathbf{u})\,\Phi\,\Pi(\mathbf{k}) = \Omega\,\Phi\,\Pi(\mathbf{h})\,\Pi(\mathbf{k})$$

$$= \Omega\,\Phi\Psi\,\Pi(\mathbf{v}) \tag{D3}$$

where, $\Pi(\mathbf{v})$ is the pure Lorentz transformation and Ψ is the rotation associated to the product $\Pi(\mathbf{h})\Pi(\mathbf{k})$ and $\mathbf{v}$ is the relative velocity resulting from $\mathbf{h}$ and $\mathbf{k}$ in analogy with (38). The relevant matrix elements of $\Pi(\mathbf{v})$ are

$$\Pi_{oo} = \gamma_h\,\gamma_k\,(1 + \mathbf{h}\cdot\mathbf{k}) \tag{D4}$$

$$\Pi_{om} = -\,\gamma_h\left[\gamma_k\,k_m + h_m + \mathbf{h}\cdot\mathbf{k}(\gamma_k - 1)k_m\,k^{-2}\right] \tag{D5}$$

$$\Pi_{mn} = \delta_{mn} + \Pi_{om}\,\Pi_{on}\,/\,\Pi_{oo} \tag{D6}$$

The associate rotation Ψ is given by

$$\Psi_{mn} = \delta_{mn} - \frac{(\gamma_h - 1)\,(\gamma_k - 1)}{\gamma_h\,\gamma_k\,(1 + \mathbf{h}\cdot\mathbf{k}) + 1}\left[\frac{k_m\,k_n}{k^2} + \frac{h_m\,h_n}{h^2}\right.$$

$$\left. -\,2\,\frac{\mathbf{h}\cdot\mathbf{k}}{h^2\,k^2}\,h_m\,k_n\right] + \frac{\gamma_h\,\gamma_k}{\gamma_h\,\gamma_k\,(1 + \mathbf{h}\cdot\mathbf{k}) + 1}\,(h_m\,k_n - k_m\,h_n) \tag{D7}$$

1 We can decompose the trasposed matrix Λ^T, which is too a Lorentz matrix. From $\Lambda^T = \Omega(\Lambda^T)\,\Pi(\Lambda^T)$, we obtain: $\Lambda = \Pi^T(\Lambda^T)\,\Omega^T(\Lambda^T) = \Pi(\Lambda^T)\,\Omega^{-1}(\Lambda^T)$.

2 $\Lambda_{oo} \leq -1$ would imply a change in the direction of time. Det $\Lambda = 1$ preserves the right-handed character of the spatial frame.

3 The quantities $-u_i$ are the components of the velocity of S, relative to S', as measured in S'.

4 From (36) it follows that $v < 1$, if $k < 1$ and $u < 1$.